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By Superdigraphs of Bipartite
Graphs**

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ON THE COMPLEXITY OF COLOURING BY SUPERDIGRAPHS OF BIPARTITE GRAPHS

by

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Abstract

Let H be a directed graph whose vertices are called colours. An H -colouring of a digraph G is an assignment of these colours to the vertices of G so that if g is adjacent to g' in G then $colour(g)$ is adjacent to $colour(g')$ in H (i.e. a homomorphism $G \rightarrow H$). In this paper we continue the study of the H -colouring problem, that is, the decision problem "Is there an H -colouring of a given digraph G ?" It follows from a result of Hell and Nešetřil that this problem is NP-complete whenever H contains a symmetric odd cycle. We consider digraphs for which the symmetric part of H is bipartite, that is, digraphs H which can be constructed from the equivalent digraph of an undirected bipartite graph by adding some arcs. We establish some sufficient conditions for these H -colouring problems to be NP-complete. A complete classification is established for the subclass of "partitionable digraphs", which we introduce.

1. Introduction.

Let D and H be directed graphs. A *homomorphism of D to H* is a function $f:V(D) \rightarrow V(H)$ such that if dd' is an arc of D , then $f(d)f(d')$ is an arc of H . When D and H are undirected graphs the definition is similar. Since an n -colouring of a graph G is a homomorphism of G to K_n , the term H -colouring has been employed to describe a homomorphism of a directed graph (resp. graph) D to a fixed directed graph (resp. graph) H . If $f(x)=h$, we sometimes say that x has been *coloured* by h . In this paper we continue the study of the H -colouring problem (H -COL); i.e. the decision problem "Does there exist an H -colouring of a given directed graph G ". Each H -colouring problem is clearly in NP.

The H -colouring problem has received considerable attention in the literature [2-5, 11, 12, 14, 17-19]. Hell and Nešetřil [14] proved that, if H is a non-bipartite undirected graph then the H -colouring problem is NP-complete, and if H is a bipartite undirected graph, then the H -colouring problem is polynomial. Subsequently attention has shifted to classifying directed H -colouring problems. There seems to be no clear distinction between the known cases of NP-complete and polynomial H -colouring problems in general (for instance there are trees T for which the problem is NP-complete [11]). For the class of connected digraphs with minimum in- and out- degree at least one, a conjecture in [3] states that H -colouring is NP-complete unless H admits a retraction (defined in section 2) to a directed cycle, and is polynomial in this case. For a summary of results in directed H -colouring, the reader should consult [2] and [17].

It follows from the result of Hell and Nešetřil [14] that $H\text{-COL}$ is NP-complete whenever H contains the equivalent digraph of a non-bipartite undirected graph. Thus in order to complete the classification of all directed graphs, it remains to consider those digraphs for which the "undirected part" is bipartite; that is, directed graphs obtained from the equivalent digraph of an undirected bipartite graph by adding some arcs. We begin by proving some general results in section three. Although these results are preliminary for section four, they are of interest in their own right as each theorem verifies the above conjecture of [3] for infinitely many new graphs. In section four we restrict our focus a little more, and introduce the family of "partitionable" digraphs, and prove that if H is a partitionable digraph, then $H\text{-COL}$ is NP-complete if H does not admit a homomorphism to the directed two-cycle C_2 , and is polynomial otherwise (cf. Theorem 4.4). This verifies the aforementioned conjecture for partitionable digraphs.

2. Preliminaries.

We use the terminology of [7]. The purpose of this section is to mention any exceptions or additions, and to review the necessary background material.

Let D be a directed graph. If both xy and yx are arcs of D , we say x and y are joined by a *double arc*, or an *undirected edge*. The *undirected part of D* , $\text{undir}(D)$, is the subdigraph of D induced by the set of double arcs. The undirected part of D is the equivalent digraph of an undirected graph, and it will often be treated as an undirected graph. We often have in mind a two-colouring of $\text{undir}(D)$, and we

sometimes talk about undirected paths or cycles in D . In our diagrams, a double arc is drawn as an undirected edge.

The directed cycle of length n is denoted by C_n , and is assumed to have vertex-set $V(C_n) = \{0, 1, \dots, n\}$ and arc-set $E(C_n) = \{01, 12, \dots, (n-2)(n-1), (n-1)0\}$. We say that a digraph is *bipartite* if it admits a C_2 -colouring.

We say that a digraph D is *connected* if any two vertices of D are joined by an oriented path. If D is not connected, we call each maximal connected subdigraph of D a *connected component* of D .

Three NP-complete problems which are used in our polynomial time transformations are described below. For more details the reader should consult [10].

k -SATISFIABILITY (k -SAT) ($k \geq 3$, fixed) [20]

INSTANCE: A set U of boolean variables, and a collection C of clauses over U such that each clause $c \in C$ involves k variables.

QUESTION: Is there a satisfying truth assignment for C ?

ONE-IN- k -SAT ($k \geq 3$, fixed)[20]

INSTANCE: A set U of boolean variables, and a collection C of clauses over U such that each clause $c \in C$ involves k variables.

QUESTION: Is there a truth assignment for U such that each clause contains precisely one true literal?

Comment: The problem remains NP-complete even if no clause contains a negated literal [15]. In this case it is the problem of two-colouring a k -regular hypergraph.

NOT-ALL-EQUAL- k -SAT ($k \geq 3$, fixed) [20]

INSTANCE: A set U of boolean variables, and a collection C of clauses over U such that each clause $c \in C$ involves k variables.

QUESTION: Is there a truth assignment for U such that each clause contains at least one true literal and at least one false literal?

Comment: The problem remains NP-complete even if no clause contains a negated literal.

Let I be a fixed digraph, and let u and v be distinct vertices of I . The *indicator construction with respect to (I, u, v)* transforms a given digraph H into the digraph H^* , defined to have the same vertex set as H , and to have as the arc set the set of all pairs hh' such that there is a homomorphism of I to H which maps u to h and v to h' . The triple (I, u, v) is called an *indicator* and, if some automorphism of I maps u to v and v to u , it is called a *symmetric indicator*. (The result of the indicator construction with respect to a symmetric indicator can be defined to be an undirected graph [14].)

2.1. Lemma [14]. If H^* -COL is NP-complete, then so is H -COL.

Suppose that H is a subdigraph of H' . A *retraction* of H' to H is a homomorphism r of H' to H such that $r(h)=h$ for all vertices h of H . A digraph is *retract-free* if it does not admit a retraction to a proper subdigraph. Every digraph H'

contains a unique (up to isomorphism) minimal retract-free subdigraph H (a *core* [3, 4, 14], or a *minimal graph* [11, 19, 21]) which admits a retraction of H' to H . If H is such a subdigraph of H' then there are homomorphisms $i: H \rightarrow H'$ (the inclusion), and $r: H' \rightarrow H$ (a retraction); thus a given digraph G is H' -colourable if and only if it is H -colourable. This allows us to restrict our attention to retract-free digraphs.

Let J be a fixed digraph, with specified vertices x and $j_1, j_2, \dots, j_t$. The *sub-indicator construction with respect to* $(J, x, j_1, j_2, \dots, j_t)$, and $h_1, h_2, \dots, h_t$ transforms a given retract-free digraph H with specified vertices $h_1, h_2, \dots, h_t$, to its subdigraph H^- induced by the vertex set V^- defined as follows: Let W be the digraph obtained from the disjoint union of J and H by identifying j_i and h_i , $i=1, 2, \dots, t$. A vertex v of H belongs to V^- just if there is a retraction of W to H which maps the vertex x to v . The tuple $(J, x, j_1, j_2, \dots, j_t)$ is called a *sub-indicator*. If each of the vertices $j_1, j_2, \dots, j_t$ is isolated, the result of the sub-indicator construction is independent of the choice of $h_1, h_2, \dots, h_t$. In this case we call $(J, x, j_1, j_2, \dots, j_t)$ a *free sub-indicator*, and in order to reflect the construction's independence of $h_1, h_2, \dots, h_t$, we denote it by (J, x, free) .

2.2. Lemma [14]. Let H be a retract-free digraph. If H^- -COL is NP-complete, then so is H -COL.

A consequence of the sub-indicator construction is the following observation.

2.3. Observation. Let D^- be the subdigraph of D induced by $V(\text{undir}(D))$. If D^- -COL is NP-complete, then so is D -COL.

Proof.

Let d be a vertex of D . The digraph D^- is the result of applying the sub-indicator construction with respect to $(C_2 \cup \{v\}, 0, v)$, and d to D . ■

We confine our attention to those digraphs D where $\text{undir}(D)$ is a spanning subdigraph. By Observation 2.3, our NP-completeness results extend to digraphs D such that $\text{undir}(D)$ is not spanning. Note, however, that it is possible for the D -colouring problem to be NP-complete even though the D^- -colouring problem (as described above) is polynomial.

A directed graph that plays a central role in this paper is *the basic triangle* C_3^* , which is constructed from an undirected path of length two, say x, z, y , by adding the arc xy , which is called *the single arc of C_3^** (cf. figure 2.1).



Figure 2.1.

Let xy be an arc of a digraph G . The arc xy is said to be *bypassed* if there is a vertex z such that xz and zy are both arcs of G . The vertex z is called a *bypass*

vertex for xy . Let B_3 be the triangle with all arcs bypassed, as shown in figure 2.2. The following results concerning superdigraphs of B_3 were proved in [5, 17].

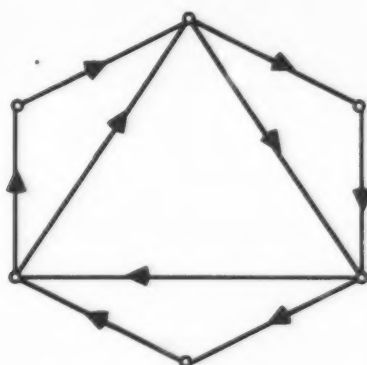


Figure 2.2.

2.4. Lemma [5, 17]. Let D be an oriented simple graph. If B_3 is a subdigraph of D , then D -COL is NP-complete.

Let T_1 be any superdigraph of B_3 , as shown in figure 2.3.

2.5. Lemma [5,17]. Let D be a digraph that contains T_1 . Then D -COL is NP-complete.

We conclude this section with another useful result from [5, 17]. Let W_3 , the wheel with three spokes, be the digraph obtained from $C_3 \cup \{v\}$ by adding the arcs $v0, v1, v2, 0v, 1v, 2v$.

2.6. Lemma [5, 17]. Let D be a digraph that contains W_3 . Then D -COL is NP-complete.

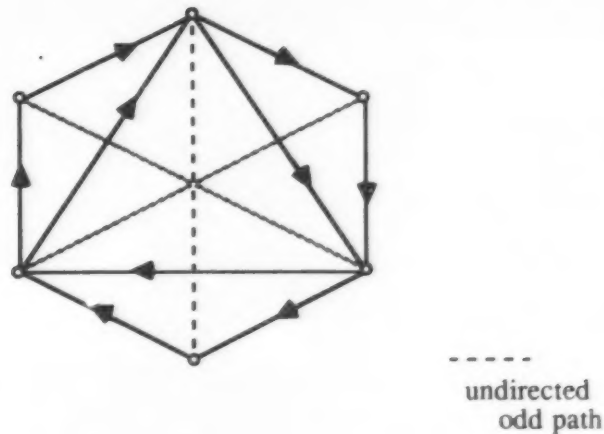


Figure 2.3.

3. Sufficient Conditions.

Let D' be a directed graph and let (R, B) be a two-colouring of $\text{undir}(D')$. We often refer to these colour classes as "red" and "blue", respectively. If D is a subdigraph of D' and all vertices of D are in the same colour class, we say that D is *monochromatic*. If it is important to distinguish which of the colour classes R and B contains the monochromatic D , we use the terms "red D " and "blue D ", respectively.

It is clear that C_2 is a retract of any bipartite digraph that contains it. Hence we may assume throughout this section that if H contains C_2 , then H is not bipartite; otherwise the H -colouring problem is polynomial (C_2 -COL is polynomial, cf. [19]). For non-bipartite digraphs we are able to prove that the problem is NP-complete in a wide variety of circumstances.

We begin by considering those digraphs D for which there exists a two-colouring of $\text{undir}(D)$ with no monochromatic C_3 or no monochromatic transitive triple.

3.1. Lemma. Let D be a digraph and C be a component of $\text{undir}(D)$. Suppose that, with respect to the unique two-colouring (R, B) of C , the $D[R]$ -colouring problem (resp. $D[B]$ -colouring problem) is NP-complete. Then D -COL is also NP-complete.

Proof.

It suffices to prove the result when the $D[R]$ -colouring problem is NP-complete. Let $v \in R$. Since C is connected, there is an (even) integer k such that for every vertex $x \in R$ there is an undirected (v, x) -walk of length k . Let P denote the equivalent digraph of an undirected path of length k , with $V(P) = \{0, 1, \dots, k\}$, and $E(P) = \{[i, i+1] : i = 0, 1, \dots, k-1\}$. The result of applying the sub-indicator construction with respect to $(P, 0, k)$, and v to D is the digraph $D[R]$. The result now follows. ■

3.2 Theorem. Let D be a digraph that contains a basic triangle. Suppose there exists a two-colouring (R, B) of $\text{undir}(D)$ such that $D[R]$ has no arcs and $D[B]$ contains no C_3 (resp. no transitive triple). Then D -COL is NP-complete.

Proof.

Suppose $D[B]$ has no C_3 (the proof being similar when $D[B]$ has no transitive triple). The transformation is from ONE-IN-THREE-SAT without negated variables. Suppose an instance of ONE-IN-THREE-SAT without negated variables is given, with variables $x_1, x_2, \dots, x_n$ and clauses $K^1, K^2, \dots, K^m$. Let X be the digraph shown in figure 3.1 with $p = 1$. Construct a digraph G from $\{x_1, x_2, \dots, x_n\}$ and m copies of the digraph X , say $X_1, X_2, \dots, X_m$, as follows. If $K^i = x_j \vee x_k \vee x_l$, identify the vertices l_1, l_2, l_3 in X_i with x_j, x_k, x_l , respectively. Clearly, the digraph G can be constructed in polynomial time.

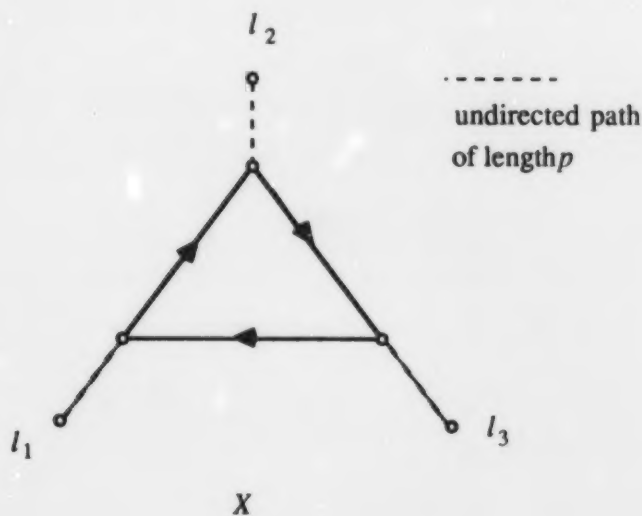


Figure 3.1

CLAIM. The digraph G is D -colourable if and only if there is a truth assignment such that each clause contains exactly one true variable.

Proof.

($\Rightarrow$) Suppose G is D -colourable. Consider a copy X_i of X . Since $D[R]$ has no arcs, at most one of l_1, l_2, l_3 is coloured by a vertex in B . Furthermore, since $D[B]$ has no C_3 , at least one of l_1, l_2, l_3 is coloured by a vertex in B . Therefore, exactly one of l_1, l_2, l_3 is coloured by a vertex in B . Define a truth assignment by setting $x_j = T$ just if x_j is coloured by a vertex in B . By the above argument, each clause contains precisely one true variable.

($\Leftarrow$) Suppose there is a truth assignment such that every clause contains exactly one true variable. Let A be a fixed basic triangle. Then A has necessarily the vertex set $\{b_1, b_2, r\}$, where $b_1, b_2 \in B, r \in R$, and single arc $b_1 b_2$. Define a partial colouring of G by assigning $colour(x_j) = b_1$ just in case $x_j = T$, and $colour(x_j) = r$ otherwise. It may be verified that, given a copy X_i of X and a partial colouring of l_1, l_2, l_3 with colours from $\{r, b_1\}$ in which exactly one of l_1, l_2, l_3 is coloured b_1 , the partial colouring can be extended to a D -colouring of X_i (in fact, to an A -colouring of X_i). Hence G is D -colourable.

The claim implies that D -COL is NP-complete. ■

3.3. Theorem. Let D be a digraph such that $undir(D)$ is a spanning subdigraph. Suppose that D contains C_3 , but no two-colouring of $undir(D)$ has a monochromatic C_3 . Then the D -colouring problem is NP-complete.

Proof.

Let (R, B) be a two-colouring of $undir(D)$. There are two cases to consider.

CASE 1. Some component of $\text{undir}(D)$ induces a subdigraph which contains three-cycles r_1, r_2, b, r_1 and b_1, b_2, r, b_1 , where $r_1, r_2 \in R$, and $b_1, b_2 \in B$.

The transformation is from NOT-ALL-EQUAL-THREE-SAT without negated variables. Suppose an instance of NOT-ALL-EQUAL-THREE-SAT without negated variables is given, with variables $x_1, x_2, \dots, x_n$ and clauses $K^1, K^2, \dots, K^m$. Let Y be the digraph shown in figure 3.1, where p is chosen to be an odd integer such that there is an undirected walk of length p from r to each of b, b_1, b_2 , and from b to each of r, r_1, r_2 . Construct a digraph G from $\{x_1, x_2, \dots, x_n\}$ and m copies of Y , say $Y_1, Y_2, \dots, Y_m$, as follows. If $K^i = x_j \vee x_k \vee x_l$, then identify vertices l_1, l_2, l_3 in Y_i with x_j, x_k, x_l , respectively. Clearly the digraph G can be constructed in polynomial time.

CLAIM. The digraph G is D -colourable if and only if there is a satisfying truth assignment in which each clause contains at least one true variable and at least one false variable.

Proof.

($\Rightarrow$) Suppose that G is D -colourable. Consider Y_i . Since D has no monochromatic C_3 , not all of the vertices l_1, l_2, l_3 can be coloured by members of R . Similarly, these vertices can not all be coloured by members of B . Thus two of l_1, l_2, l_3 are coloured by members of R and the remaining one is coloured by a vertex in B , or vice-versa. Define a truth assignment by setting $x_j = T$ just if vertex x_j is coloured by a member of R . By the above argument, every clause contains a true variable and a false variable.

($\Leftarrow$) Suppose there is a truth assignment such that every clause contains at least one true variable, and at least one false variable. Define a partial

colouring of G by assigning $colour(x_j)=b$ just in case $x_j=T$, and $colour(x_j)=r$ otherwise. It may be directly verified that, given a copy Y_i of Y and a partial colouring of l_1, l_2, l_3 with colours from $\{r, b\}$ in which at least one, but not all of l_1, l_2, l_3 , is coloured b , the partial colouring can be extended to a D -colouring of Y_i . Hence G is D -colourable.

This completes the proof of case 1.

CASE 2. The subdigraph induced by every component of $undir(D)$ contains only three-cycles with two vertices from R and one vertex from B , or vice-versa.

By switching colours in some components of $undir(D)$, it may be assumed that every three-cycle has two vertices in R and one vertex in B . Let $A = r_1, r_2, b$ be a fixed copy of C_3 in the subdigraph induced by some component of $undir(D)$, where $r_1, r_2 \in R$ and $b \in B$. Let Y be the digraph shown in figure 3.1, and choose p such that there is an undirected walk of length p from b to r_1 and to r_2 . The transformation is once again from ONE-IN-THREE-SAT without negated variables. Suppose an instance of ONE-IN-THREE-SAT without negated variables is given, with variables $x_1, x_2, \dots, x_n$ and clauses $K^1, K^2, \dots, K^m$. Construct a digraph G from $\{x_1, x_2, \dots, x_n\}$ and m copies of Y , say $Y_1, Y_2, \dots, Y_m$, as follows. If $K^i = x_j \vee x_k \vee x_l$, then identify vertices l_1, l_2, l_3 in Y_i with x_j, x_k, x_l , respectively. Clearly, the digraph G can be constructed in polynomial time.

CLAIM The digraph G is D -colourable if and only if there is a truth assignment such that each clause contains exactly one true variable.

Proof.

($\Rightarrow$) Suppose G is D -colourable. Consider a copy Y_i of Y . Since $D[R]$ has no three-cycle, at least one of l_1, l_2, l_3 is coloured by a vertex in B . Since $D[B]$ has no three-cycle, at least one of l_1, l_2, l_3 is coloured by a vertex in B . Moreover, since there is no three-cycle which has two vertices in B , exactly one of l_1, l_2, l_3 is coloured by a vertex in B . Define a truth assignment by setting $x_j = T$ just if x_j is coloured by a vertex in B . By the above argument, each clause contains precisely one true variable.

($\Leftarrow$) Suppose there is a truth assignment such that every clause contains exactly one variable with the value T . Define a partial colouring of G by assigning $colour(x_j) = r_1$ just in case $x_j = T$, and $colour(x_j) = b$ otherwise. It may be directly verified that, given a copy Y_i of Y , and a partial colouring of l_1, l_2, l_3 with colours from $\{r_1, b\}$ in which exactly one of l_1, l_2, l_3 is coloured r_1 , the partial colouring can be extended to a D -colouring of Y_i . Hence G is D -colourable.

This completes the proof of case 2. ■

The next result generalises Theorem 3.2.

3.4. Theorem. Let D be a digraph such that $undir(D)$ is a spanning subdigraph. Suppose that D contains a basic triangle. If D does not contain a monochromatic transitive triple, then the D -colouring problem is NP-complete.

Proof.

The proof is similar to the proof of Theorem 3.3. The component that corresponds to each clause is shown in figure 3.2. The details are left to the reader. ■

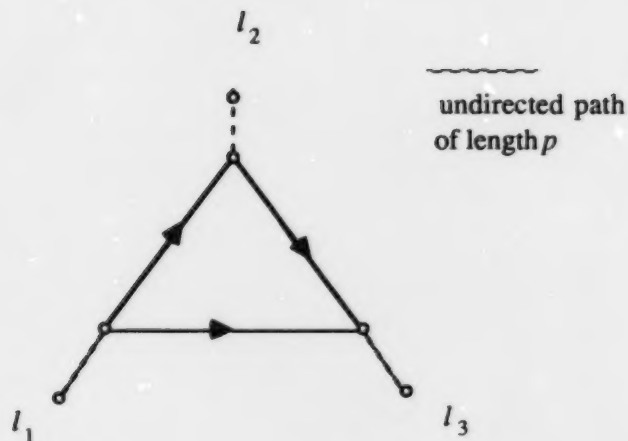


Figure 3.2

We now turn our attention to superdigraphs of undirected bipartite graphs that contain the digraph B_3 , the directed triangle with all arcs bypassed, defined in section two (cf. figure 2.2) We show that $H\text{-COL}$ is NP-complete for any superdigraph H of B_3 with the property that $\text{undir}(H)$ is spanning and connected (Theorem 3.7).

3.5. Lemma. Let D be a digraph such that $\text{undir}(D)$ is a connected spanning subdigraph. Suppose there exists a two-colouring (R, B) of $\text{undir}(D)$ such that there is a monochromatic C_3 in which all arcs are bypassed. If the bypass vertices x_1, x_2, x_3 are all in the same colour class, then $D\text{-COL}$ is NP-complete.

Proof.

Without loss of generality assume the C_3 in the statement of the theorem is red. If $x_1, x_2, x_3 \in R$, then the $D[R]$ -colouring problem is NP-complete by Lemmas 3.1 and 2.4. Consequently the D -colouring problem is also NP-complete. Suppose that x_1, x_2, x_3 are in B . Then the D -colouring problem is NP-complete by Lemma 2.5 (each x_i is joined to every vertex on the three-cycle by an undirected odd path because $\text{undir}(G)$ is spanning and connected). ■

3.6.Theorem. Let D be a digraph such that $\text{undir}(D)$ is spanning and connected. Suppose that (R,B) is a two-colouring of $\text{undir}(D)$ such that there is a monochromatic C_3 in which all arcs are bypassed. Then $D\text{-COL}$ is NP-complete.

Proof.

There are three cases to consider.

CASE 1. There is a monochromatic C_3 in which all arcs are bypassed, and the bypass vertices are all in the same colour class.

Then $D\text{-COL}$ is NP-complete by Lemma 3.5.

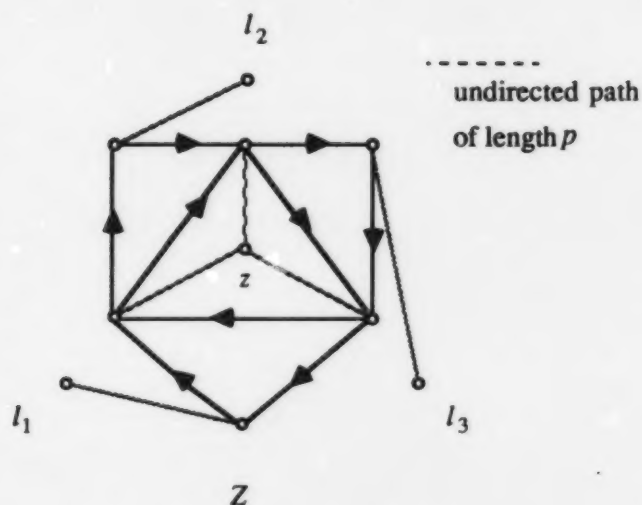


Figure 3.3.

CASE 2. There are monochromatic three-cycles C, C' with bypass vertices a, b, c , and a', b', c' , respectively, where $a, b, c' \in R$, and $a', b', c \in B$.

It may be assumed that case 1 does not hold simultaneously. The transformation is from NOT-ALL-EQUAL-THREE-SAT without negated variables. Let Z be the digraph shown in figure 3.3, where p is chosen so that any two vertices in the same colour class are joined by an undirected walk of length p (the (even) length p exists because $\text{undir}(D)$ is spanning and connected). Let an instance of NOT-ALL-EQUAL-THREE-SAT without negated variables be given, with variables $x_1, x_2, \dots, x_n$ and clauses $K^1, K^2, \dots, K^m$. Construct a digraph G from $\{x_1, x_2, \dots, x_n\}$ and m copies of Z , say $Z_1, Z_2, \dots, Z_m$, as follows. Suppose $K_i = x_j \vee x_k \vee x_l$. Then identify the vertices l_1, l_2, l_3 in Z_i with x_j, x_k, x_l , respectively. Clearly, the digraph G can be constructed in polynomial time.

CLAIM. The digraph G is D -colourable if and only if there is a satisfying truth assignment in which each clause has at least one true variable, and at least one false variable.

Proof.

Consider a D -colouring of G . It is not hard to see that the three-cycle in each copy of Z must be monochromatic (it must map into the same colour class as z). The remaining details are similar to the proof of the analogous claim in Theorem 3.3, and are left to the reader.

This completes the proof of case 2.

CASE 3. Every monochromatic C_3 with bypasses on all arcs has red two bypass vertices and one blue bypass vertex, or vice-versa.

The transformation is from ONE-IN-THREE-SAT without negated variables. The construction of the digraph G is identical to case 2. The remaining details are similar to those in the analogous case of Theorem 3.3, and are left to the reader.

All three cases have been considered, and the result follows. ■

3.7. Theorem. Let D be a digraph such that $\text{undir}(D)$ is a connected spanning subdigraph. If D contains B_3 , then the D -colouring problem is NP-complete.

Proof.

By Theorem 3.6, it suffices to consider the case when no copy of B_3 in D contains a monochromatic C_3 . The proof is identical to the proof of Theorem 3.3,

except that bypasses must be added to all arcs of the three cycle in the digraph in figure 3.1. The details may be easily supplied by the reader. ■

Let $n > 1$, and let F_n , the n -fan, be the digraph with vertex set $\{0, 1, \dots, n\}$, and arc set $\{01, 10, 02, 20, \dots, 0n, n0\} \cup \{12, 23, \dots, (n-1)n\}$. The following corollary is useful in the next section.

3.8. Corollary. For any $n > 3$, if H is a superdigraph of F_n such that $\text{undir}(H)$ is spanning and connected, then $H\text{-COL}$ is NP-complete.

Proof.

If $n > 3$, then F_n contains B_3 . ■

4. Partitionable digraphs.

In this section we introduce a subclass of superdigraphs of bipartite graphs which we call "partitionable digraphs". We give a complete classification by complexity of the digraphs in this class.

Throughout this section, a *single arc* of a digraph D is an arc $xy \in E(D)$ such that $yx \notin E(D)$.

We say that a digraph D is *partitionable* if the following two conditions are satisfied:

- (i) $\text{undir}(D)$ is a spanning subdigraph of D , and
- (ii) there is a two-colouring of $\text{undir}(D)$, such that every single arc is monochromatic.

Another way to express condition (ii) is the following: There exists a two-colouring of $\text{undir}(D)$ such that all double arcs are between the colour classes and all single arcs are within the colour classes. Partitionable digraphs are precisely the digraphs obtained from (the equivalent digraph of) an undirected bipartite graph by adding monochromatic arcs.

Let D be a partitionable digraph. We have previously noted that D may be assumed to be non-bipartite, otherwise $D\text{-COL}$ is polynomial (cf. the beginning of section three). In the remainder of this section we show that the D -colouring problem is NP-complete for all non-bipartite partitionable digraphs D . In particular, we prove the following theorem.

4.1. Theorem. Let D be a partitionable digraph. If D is bipartite, then $D\text{-COL}$ is polynomial. Otherwise (D contains an oriented odd cycle), $D\text{-COL}$ is NP-complete.

The proof requires some preliminary lemmas. Let T_2 be the digraph shown in figure 4.1 (a).

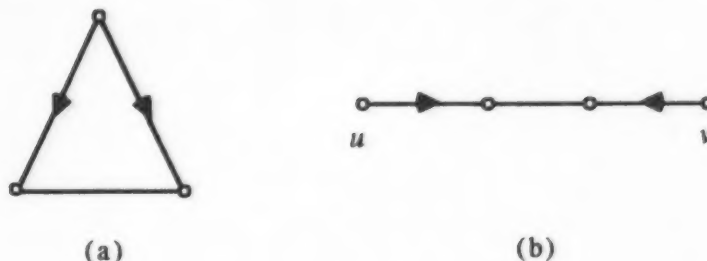


Figure 4.1.

4.2. Lemma. Let D be a digraph for which $\text{undir}(D)$ is a spanning subdigraph, and suppose T_2 is not a subdigraph of D . If D is not bipartite, then $D\text{-COL}$ is NP-complete.

Proof.

Assume D does not contain T_2 . Let (I, u, v) be the indicator shown in figure 4.1 (b). Let D^* be the result of applying the indicator construction with respect to (I, u, v) to D . Then D^* is loopless and undirected. Since $\text{undir}(D)$ is spanning, D^* is the equivalent digraph of the underlying graph corresponding to D . Since D is not an orientation of a bipartite graph, D^* has an odd cycle, whence the D^* -colouring problem is NP-complete. Therefore $D\text{-COL}$ is also NP-complete. ■

4.3. Corollary. If D is a non-bipartite partitionable digraph that does not contain a basic triangle, then the D -colouring problem is NP-complete. ■

Let (J_1, x, z) and (J_2, x, z) be the sub-indicators shown in figure 4.2. These are important in the proof of Theorem 4.1.

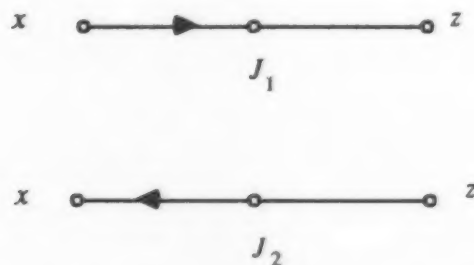


Figure 4.2.

4.4. Lemma. Let D be a partitionable digraph, and let (R, B) be a two-colouring of $\text{undir}(D)$. Let $r \in R$, and let $i \in \{1, 2\}$. Let D^- be the result of applying the sub-indicator construction with respect to (J_i, x, z) , and r , to D . If r' is a red vertex that belongs to $V(D^-) - r$, then there is an undirected path of length two in D joining r and r' .

Proof.

This follows immediately from the definition of a partitionable digraph. ■

There are obviously other versions of Lemma 4.4 corresponding to the possible cases that arise when the sub-indicator is (J_i, z, x) , or the construction takes place with respect to a vertex in B . For brevity, these are not stated, but they are used.

Proof of Theorem 4.1.

It remains to show that if D is a partitionable digraph that contains a basic triangle, then the D -colouring problem is NP-complete. Suppose that D is a counterexample with the minimum number of vertices. That is, D is a partitionable digraph that contains a basic triangle, with the minimum number of vertices such that the D -colouring problem is not NP-complete.

We would like to use the sub-indicator construction to deduce various structural properties of D . There is, unfortunately, a difficulty: Let D^- be the result of applying a sub-indicator construction to D . The digraph $\text{undir}(D^-)$ may not be spanning, consequently D^- may not be partitionable. This is not a severe handicap, as the following fact (*) turns out to be sufficient.

(*) Suppose that the result D^- of applying a sub-indicator construction to D contains a basic triangle. Then $D = D^-$.

Proof of (*).

Let D^{--} be the result of applying the sub-indicator construction with respect to $(C_2, 0, \text{free})$ to D^- . Then D^{--} is a partitionable digraph (it is an induced subdigraph of D , and, by the construction, $\text{undir}(D^{--})$ is a spanning subdigraph) that contains a basic triangle. Suppose D^{--} is a proper subdigraph of D . Then our choice of D implies that the D^{--} -colouring problem is NP-complete. Hence the D -colouring problem is also NP-complete, a contradiction. Therefore $D = D^-$.

The theorem follows from the sequence of claims below in which we use (*) to derive some structural properties of D , and ultimately a contradiction.

(1) The digraph $\text{undir}(D)$ is connected.

Let C be a fixed basic triangle. Without loss of generality, assume that the single arc pq of C is red. Let b be the third vertex of C . Let D^- be the result of applying the sub-indicator construction with respect to (J_1, z, x) , and r to D . Then D^- contains a basic triangle and by (*), $D = D^-$. Hence, by Lemma 4.4, there is in D an undirected path of length two from r to any other red vertex. Therefore there is also an undirected path of length one or three from r to any blue vertex. This completes the proof of (1).

The next two claims concern forbidden subdigraphs of D . Recall the digraph F_n defined in section three and the digraph W_3 defined in section two.

(2) The digraph D contains neither F_4 nor W_3 .

In either case $D\text{-COL}$ is NP-complete, a contradiction (cf. Corollary 3.8 and Lemma 2.6, respectively).

(3) The digraph D does not contain F_3 .

Suppose D contains a copy of F_3 labelled as shown in figure 4.3. Since D contains neither F_4 nor W_3 , there is no vertex p such that $pr, [p, q] \in E(D)$. Let D^- be the result of applying the indicator construction with respect to (J_2, x, z) , and q to D . It is not hard to check that $q, s, t \in V(D^-)$, but $r \notin V(D^-)$. Hence D^- contains a basic triangle and has fewer vertices than D . By our choice of D , the D^- -colouring problem is NP-complete. Therefore the D -colouring problem is also NP-complete, which is a contradiction.

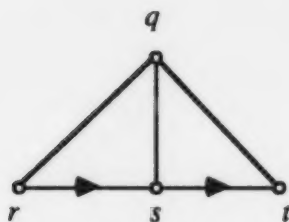


Figure 4.3.

(4) If ab is a single arc in a basic triangle of D , then there is no d (resp. x) in $V(D)$ such that bd (resp. xa) is also a single arc.

We prove only that the vertex d can not exist. Let c be the third vertex of the basic triangle. Suppose to the contrary that d exists. Let D^- be the result of applying the sub-indicator construction with respect to (J_2, z, x) , and b to D . Since D^- contains the basic triangle induced by $\{a, b, c\}$, we have by (*) that $D = D^-$. Since D is partitionable, this implies that there exists a vertex e and double arcs $[b, e]$ and $[d, e]$. By (3), $e \neq c$. Let D^{--} be the result of applying the sub-indicator construction with respect to (J_2, x, z) and c to D^- . Then $b, d, e \in V(D^{--})$, but since D^- does not contain F_3 , $a \notin V(D^{--})$. Hence D^{--} contains a basic triangle (induced by $\{b, d, e\}$), and has fewer vertices than D . By our choice of D , the D^- -colouring problem is NP-complete. Therefore D -COL is also NP-complete. This contradiction completes the proof of (4).

We define a *red basic triangle* (resp. *blue basic triangle*) to be a basic triangle in which the single arc is red (resp. blue).

(5) Let $r \in R$. Then there is a red basic triangle C , with $V(C) = \{r_1, r_2, b\}$, and $r_1 r_2 \in R$, such that $[r, b]$ is a double arc.

The statement is true if r is in a red basic triangle. Suppose that r is not in a red basic triangle. Let C' be a red basic triangle and let $V(C') = \{r', r'', b'\}$, with single arc $r' r''$. Let D^- be the result of applying the sub-indicator construction with respect to (J_1, z, x) and r' to D . Then D^- contains a basic triangle, so by (*), $D = D^-$ and hence r and r' are joined by an undirected path of length two, say r', b, r . If b and r'' are adjacent the proof is complete. Suppose that b and r'' are not adjacent. Since b is a vertex of D^- , the definition of $V(D^-)$ implies that there exists a vertex r''' , an arc $r' r'''$, and a double arc $[r''', b]$. The three-set $\{r', r''', b\}$ induces the desired basic triangle.

(6) Let $b \in B$. Then there is a blue basic triangle C , with $V(C) = \{b_1, b_2, r\}$, and $b_1 b_2 \in B$ such that $[b, r]$ is a double arc.

The proof is similar to (5).

(7) If D has a red basic triangle (resp. blue basic triangle) then $D[R]$ (resp. $D[B]$) has no directed path of length two.

We prove the result on the assumption that there is a red basic triangle (the proof of the other case being similar). Suppose to the contrary that D has a red basic triangle and $D[R]$ has a directed path of length two, say r, r', r'' . By (5) there exists a red basic triangle C , with vertex set $\{u, v, b\}$, $b \in B$, single arc uv , and such that there is a double arc $[r, b]$. Let D^- and D^{--} denote the result of applying the sub-indicator construction with respect to (J_1, z, x) and r_1 , and (J_2, z, x) and r_2 to D , respectively. Since both D^- and D^{--} contain the basic triangle C , (*) implies that

$D = D^- = D^{--}$. By Lemma 4.4 there is an undirected path of length two between u and r' , and also between v and r' . Let D^{---} be the result of applying the sub-indicator construction with respect to (J_2, z, x) and r' to D . Then D^{---} contains the basic triangle C , so (*) implies that $D^{---} = D$. Hence there is also an undirected path of length two joining r'' and r'' , say r', b', r'' . Thus $\{r', r'', b\}$ induces a basic triangle. But $rr' \in E(D)$, contrary to (4). This completes the proof of (7).

(8) Both $D[R]$ and $D[B]$ have arcs.

If, say, $D[R]$ has no arcs, then D satisfies the hypotheses of Theorem 3.2; it contains a basic triangle and, since $D[B]$ has no directed path of length two, it does not contain a transitive triple. Hence the D -colouring problem is NP-complete, which is a contradiction.

(9) Suppose D contains a red (resp. blue) basic triangle. Then every vertex in B (resp. R) is in a red (resp. blue) basic triangle.

Suppose that $\{r_1, r_2, b'\}$, $b' \in B$, induces a red basic triangle. Let $b \in B$. Let D^- , and D^{--} be the result of applying the sub-indicator construction with respect to (J_1, z, x) and r_1 , and (J_2, z, x) and r_2 to D , respectively. Since both $V(D^-)$ and $V(D^{--})$ contain the basic triangle induced by $\{r_1, r_2, b'\}$, (*) implies that $D = D^- = D^{--}$. Thus there exists a vertex r' such that $r_1 r'$ is an arc and $[r', b]$ is a double arc. Moreover there is an undirected path of length two joining r_1 and r' , and another such path joining r_2 and r' . Let D^{---} be the result of applying the sub-indicator construction with respect to (J_2, z, x) and r' to D . Since $r_1, r_2, b \in V(D^{---})$, the digraph D^{---} contains a basic triangle, hence (*) implies that $D^{---} = D$. Therefore there exists a vertex r'' such that $r'' r'$ is an arc and r'' is adjacent to b via a double arc. But now $\{r', r'', b\}$ induces a red basic triangle. This completes the proof of (9).

(10) The digraph D contains a red basic triangle and a blue basic triangle.

Without loss of generality, suppose there is a red basic triangle. Let bb' be an arc of $D[B]$ (the existence of such an arc is guaranteed by (8)). By (9), every vertex in B is in a red basic triangle. Suppose $\{r_1, r_2, b\}$ and $\{r', r'', b'\}$ each induce a basic triangle, with single arcs $r_1 r_2$ and $r' r''$, respectively. Let D^- be the result of

applying the sub-indicator construction with respect to (J_1, z, x) and r_1 to D . Since $r_1, r_2, b \in V(D^-)$, $(*)$ asserts that $D=D^-$. Consequently there is an undirected path of length two joining r_1 and r' , and another such path between r_1 and r'' . Let D^{--} be the result of applying the sub-indicator construction with respect to (J_2, x, z) and r_1 to $D (=D^-)$. Then $r', r'', b \in V(D^{--})$, so $(*)$ implies that $D=D^{--}$. Since $b \in V(D^{--})$, there is a vertex b_1 such that b_1b is an arc, and $[b_1, r_1]$ is a double arc. The three-set $\{b_1, b, r_1\}$ induces a blue basic triangle.

By (10), D has no monochromatic directed path of length two. Hence D satisfies the hypotheses of Theorem 3.3, leading to the contradiction that the D -colouring problem is NP-complete. This completes the proof. ■

As a corollary we derive an NP-completeness result for a class of digraphs that are not partitionable, and that do not necessarily have the property that $\text{undir}(D)$ is spanning. Let $0 \leq a_1 < a_2 < \dots < a_t < n$ be integers and $PW(a_1, a_2, \dots, a_t)$ be the digraph constructed from $C_n \cup \{v\}$ by adding the double arcs $\{[v, a_i] : i=1, 2, \dots, t\}$. We call $PW(a_1, a_2, \dots, a_t)$ a *partial wheel*.

4.5. Corollary. If $PW(a_1, a_2, \dots, a_t)$ is bipartite, then $PW(a_1, a_2, \dots, a_t)$ -COL is polynomial. Otherwise $PW(a_1, a_2, \dots, a_t)$ -COL is NP-complete.

Proof.

We have previously noted the first statement (cf. the beginning of section three). Suppose $PW(a_1, a_2, \dots, a_t)$ is not bipartite. Then there exists i such that $a_{i+1}-a_i$ is odd. Let $k = \min \{a_{j+1}-a_j : 1 \leq j \leq t \text{ and } a_{j+1}-a_j \text{ is odd}\}$. Let (I, u, v) be the

indicator constructed by adjoining two-cycles to the endvertices u and v of a directed path of length k . Let H^* be the result of applying the indicator construction with respect to (I, u, v) to H . If $n=2k$, the digraph H^* is an undirected triangle. Otherwise, the core of H^* is a partitionable digraph that contains a basic triangle. In both cases H^* -COL is NP-complete. Therefore H -COL is also NP-complete. ■

5. Conclusion.

In this paper we have described a variety of sufficient conditions for NP-completeness of the D -colouring problem when D is a superdigraph of an undirected bipartite graph. Based on our results, we make the following conjecture.

5.1. Conjecture. Let D be a superdigraph of an undirected bipartite graph, and suppose that $\text{undir}(D)$ is a spanning subdigraph. If D is not bipartite, then D -COL is NP-complete. Otherwise D -COL is polynomial.

We note that this is a special case of Conjecture 6.1 in [3]; nevertheless, even this special case seems difficult. As a next step, one might try to show that it is true in the presence of the additional condition " $\text{undir}(D)$ is connected". By using the indicator construction in a manner similar to [14], it can be shown that it would suffice to prove this for digraphs that contain a basic triangle. Other results in this paper can be used to deduce a variety of structural properties of a hypothetical counterexample to Conjecture 5.1.

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